Neural Stochastic Contraction Metrics for Robust Control and Estimation

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Abstract—**We present neural stochastic contraction metrics, a new design framework for provably-stable robust control and estimation for a class of stochastic nonlinear systems. It exploits a spectrally-normalized deep neural network to construct a contraction metric, sampled via simplified convex optimization in the stochastic setting. Spectral normalization constrains the state-derivatives of the metric to be Lipschitz continuous, and thereby ensures exponential boundedness of the mean squared distance of system trajectories under stochastic disturbances. This framework allows autonomous agents to approximate optimal stable control and estimation policies in real-time, and outperforms existing nonlinear control and estimation techniques including the state-dependent Riccati equation, iterative LQR, EKF, and deterministic neural contraction metric method, as illustrated in simulations.**

Index Terms—**Machine learning, Stochastic optimal control, Observers for nonlinear systems.**

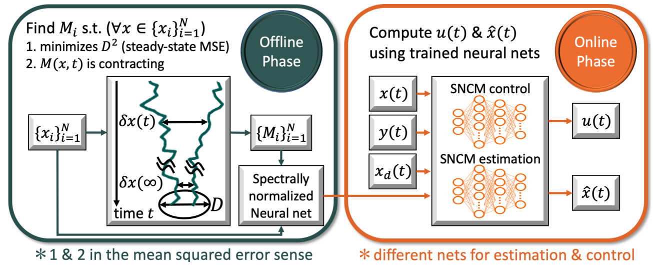

Fig. 1. Illustration of NSCM: $M(x,t)$ denotes optimal contraction metric; x_i and M_i denote sampled states and contraction metrics; $y(t)$ denotes measurements; $x(t)$, $x_d(t)$, and $\hat{x}(t)$ denote actual, target, and estimated trajectories, respectively.

I. Introduction

A Key challenge in designing autonomous aerospace robots operating in real-time is how we assure optimality and stability of their control and estimation schemes even for nonlinear systems with unbounded stochastic disturbances, the time evolution of which is expressed as Itô stochastic differential equations [1]. As their onboard computational power is often limited, it is desirable to execute control and estimation policies computationally as cheaply as possible.

In this paper, we present a Neural Stochastic Contraction Metric (NSCM) based robust control and estimation framework outlined in Fig. 1. It uses a spectrally-normalized neural network as a model for an optimal contraction metric (differential Lyapunov function), the existence of which guarantees exponential boundedness of the mean squared distance between two system trajectories perturbed by stochastic disturbances. Unlike the Neural Contraction Metric (NCM) [2], where we proposed a learning-based construction of optimal contraction metrics for control and estimation of nonlinear systems with bounded disturbances, stochastic contraction theory [3]–[5] guarantees stability and optimality in the mean squared error sense for unbounded stochastic disturbances via convex optimization. We then introduce Spectral Normalization (SN) [6] in the NSCM training, in order to validate a major assumption in stochastic contraction that the first state-derivatives of the metric are Lipschitz. We also extend the State-Dependent-Coefficient (SDC) technique [7] further to include a target trajectory in control and estimation, for the sake of global exponential stability of unperturbed systems.

In the offline phase, we sample contraction metrics by solving convex optimization to minimize an upper bound of the steady-state mean squared distance of stochastically perturbed system trajectories (see Fig. 1). Other convex objectives such as control effort could be used depending on the application of interest. We call this method the modified CV-STEM (mCV-STEM), which differs from the original [8] in the following points: 1) a simpler stochastic contraction condition with an affine objective function both in control and estimation, thanks to the Lipschitz condition on the first derivatives of the metrics; 2) generalized SDC parameterization (i.e. A given by $A(x,x_d,t)(x-x_d) = f(x,t)-f(x_d,t)$ instead of $A(x,t) = f(x,t)x$) which results in global exponential stability of unperturbed systems even with a target trajectory, x_d for control and x for estimation; and 3) optimality in the contraction rate α and design parameter ε. The second point is also applicable to deterministic settings including the NCM [2]. We remark that reference-independent integral forms of control laws [9]–[13] could be considered by changing how we sample the metrics in this phase. Also, its contraction theory-based formulation enables the use of combination properties for preserving contraction of systems with time-delayed communications [14] or hierarchical combination [15] (e.g. output feedback control). We train a neural network with the sampled metrics subject to the aforementioned Lipschitz constraint using the SN technique.

In the online phase, the trained NSCM models are exploited to approximate the optimal control and estimation policies, which only require one neural network evaluation at each time step as shown in Fig 1. The benefits of this framework are demonstrated in the rocket state estimation and control problem, by comparing it with the State-Dependent Riccati

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Code: https://github.com/astrohiro/nscm.

Equation (SDRE) method [5], [7], Iterative LQR (ILQR) [16], [17], EKF, NCM, and mCV-STEM.

Related Work: Contraction theory [15] is an analytical tool for studying the differential dynamics of a nonlinear system under a contraction metric, whose existence leads to a necessary and sufficient characterization of its exponential incremental stability. The theoretical foundation of this paper rests on its extension to stability analysis of stochastic nonlinear systems [3]–[5]. The major difficulty in applying it in practice is the lack of general analytical schemes to obtain a suitable stochastic contraction metric for nonlinear systems written as Itô stochastic differential equations [1].

For deterministic nonlinear systems, there are several machine learning-based techniques for designing real-time computable optimal Lyapunov functions/contraction metrics. These include [2], [18], [19], where neural networks are used to represent solutions to their respective optimization problem for obtaining a Lyapunov function. This paper is distinct from our previous study [2], as the NSCM explicitly considers the case of stochastic nonlinear systems, where deterministic control and estimation policies could fail due to additional terms in the differential of a given function under stochastic perturbation involving its second derivatives.

The CV-STEM method [8] is proposed to construct a contraction metric explicitly accounting for the stochastic aspects of dynamical processes. It is designed to minimize an upper bound of the steady-state mean squared tracking error of stochastic nonlinear systems, assuming that the first and second derivatives of the metric with respect to their state variables are bounded. In this paper, instead, we only assume that the first derivatives are Lipschitz continuous, thereby enabling the use of spectrally-normalized neural networks [6] as explained earlier. This also significantly reduces the computational burden in solving the CV-STEM optimization problems, allowing autonomous agents to perform both optimal control and estimation tasks in real-time.

II. Preliminaries

We use $\|x\|$ and $\|A\|$ for the Euclidean and induced 2-norm, and $A \succ 0$, $A \succeq 0$, $A \prec 0$, and $A \preceq 0$ for positive definite, positive semi-definite, negative definite, and negative semi-definite matrices, respectively. Also, $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix, $E[\cdot]$ denotes the expected value, f_x denotes the partial derivative of $f(x)$, and $\mathrm{sym}(A) = (A+A^T)/2$.

A. Neural Network and Spectral Normalization

A neural network is a mathematical model for representing training samples $\{(x_i,y_i)\}_{i=1}^N$ of $y=\phi(x)$ by optimally tuning its hyperparameters W_ℓ, and is given as

$$y_i = \varphi(x_i) = T_{L+1} * \sigma * T_L * \cdots * \sigma * T_1(x_i) \tag{1}$$

where $T_\ell(x) = W_\ell x$, $*$ denotes convolution, and σ is an activation function $\sigma(x) = \tanh(x)$. Note that $\varphi(x) \in C^\infty$.

Spectral normalization (SN) [6] is a technique to overcome the instability of neural network training by constraining (1) to be globally Lipschitz, i.e., $\exists\ L_{nn} \geq 0$ s.t. $\|\varphi(x)-\varphi(x')\| \leq L_{nn}\|x-x'\|,\ \forall x,x'$, which is shown to be useful in nonlinear control designs [20]. SN normalizes the weight matrices W_ℓ as $W_\ell = (C_{nn}\Omega_\ell)/\|\Omega_\ell\|$ with $C_{nn} \geq 0$ being a given constant, and trains a network with respect to Ω_ℓ. Since this results in $\|\varphi(x)-\varphi(x')\| \leq C_{nn}^{L+1}\ \|x-x'\|$ [6], selecting C_{nn} as $L_{nn}^{1/(L+1)}$ guarantees Lipschitz continuity of $\varphi(x)$. In Sec. III-B, we propose one way to train a neural network that guarantees the Lipschitz continuity assumption on M_{x_i} in Theorem 1, using the SN technique introduced here.

B. Stochastic Contraction Analysis for Incremental Stability

Consider the following nonlinear system with stochastic perturbation given by the Itô stochastic differential equation:

$$dx = f(x,t)dt + G(x,t)d\mathscr{W}(t),\ x(0)=x_0 \tag{2}$$

where $t \in \mathbb{R}_{\geq 0}$, $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $G : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times d}$, $\mathscr{W}(t)$ is a d-dimensional Wiener process, and x_0 is a random variable independent of $\mathscr{W}(t)$ [21]. We assume that 1) $\exists L_1 > 0$ s.t. $\|f(x_1,t)-f(x_2,t)\| + \|G(x_1,t)-G(x_2,t)\|_F \leq L_1\|x_1-x_2\|$, $\forall t \in \mathbb{R}_{\geq 0}$ and $\forall x_1,\ x_2 \in \mathbb{R}^n$, and 2) $\exists L_2 > 0$, s.t. $\|f(x_1,t)\|^2 + \|G(x_1,t)\|_F^2 \leq L_2(1+\|x_1\|^2)$, $\forall t \in \mathbb{R}_{\geq 0}$ and $\forall x_1 \in \mathbb{R}^n$ for the sake of existence and uniqueness of the solution to (2).

Theorem 1 analyzes stochastic incremental stability of two trajectories of (2), x_1 and x_2. In Sec. IV, we use it to find a contraction metric $M(x,t)$ for given α, ε, and L_m, where α is the contraction rate, ε is an new parameter for disturbance attenuation (see Theorem 1), and L_m is the Lipschitz constant of M_{x_i}. Note that ε and L_m are introduced for the sake of stochastic contraction and were not present in the deterministic case [2]. Sec. IV-B2 delineates how we select them in practice.

Theorem 1: Suppose $\exists g_1, g_2 \in [0,\infty)$ s.t. $\|G(x_1,t)\|_F \leq g_1$ and $\|G(x_2,t)\|_F \leq g_2$, $\forall x,t$. Suppose also that $\exists M(x,t) \succ 0$ s.t. M_{x_i}, $\forall i$ is Lipschitz continuous, i.e. $\|M_{x_i}(x,t)-M_{x_i}(x',t)\| \leq L_m\|x-x'\|$, $\forall x,x',t$ with $L_m \geq 0$. If $M(x,t) \succ 0$ and $\alpha,\varepsilon,\underline{\omega},\overline{\omega} \in (0,\infty)$ are given by

$$\dot{M}(x,t) + 2\,\mathrm{sym}\left(M(x,t)f_x(x,t)\right) + \alpha_g I \preceq -2\alpha M(x,t) \tag{3}$$

$$\overline{\omega}^{-1} I \preceq M(x,t) \preceq \underline{\omega}^{-1} I,\ \forall x,t \tag{4}$$

where $\alpha_g = L_m(g_1^2+g_2^2)(\varepsilon+1/2)$, then the mean squared distance between x_1 and x_2 is bounded as follows:

$$E\left[\|x_1-x_2\|^2\right] \leq \frac{C}{2\alpha}\frac{\overline{\omega}}{\underline{\omega}} + \overline{\omega}E[V(x(0),\delta x(0),0)]e^{-2\alpha t}. \tag{5}$$

where $V(x,\delta x,t) = \delta x^T M(x,t)\delta x$ and $C = (g_1^2+g_2^2)(2/\varepsilon+1)$.

Proof: Let us first derive the bounds of M_{x_i} and $M_{x_ix_j}$. Since M_{x_i}, $\forall i$ is Lipschitz, we have $\|M_{x_ix_j}\| \leq L_m$, $\forall i,j$ by definition. For $h \geq 0$ and a unit vector e_i with 1 in its ith element, the Taylor's theorem suggests $\exists \xi_-, \xi_+ \in \mathbb{R}^n$ s.t.

$$M(x \pm he_i,t) = M(x,t) \pm M_{x_i}(x,t)h + M_{x_ix_i}(\xi_\pm,t)h^2/2. \tag{6}$$

This implies that $\|M_{x_i}\|$ is bounded as $\|M_{x_i}\| \leq h^{-1}\underline{\omega}^{-1} + L_m h/2 \leq \sqrt{2L_m\underline{\omega}^{-1}}$. Note that $h = \sqrt{2/(L_m\underline{\omega})}$ is substituted to obtain the last inequality. Next, let $\mathscr{L}$ be the infinitesimal differential generator given in [8]. Computing $\mathscr{L}V$ using these bounds as in [8] yields

$$\begin{aligned}\mathscr{L}V \leq&\delta x^T\left(\dot{M}+2\,\mathrm{sym}\left(Mf_x\right)\right)\delta x\\&+(g_1^2+g_2^2)(L_m\|\delta x\|^2/2+2\sqrt{2L_m\underline{\omega}^{-1}}\|\delta x\|+\underline{\omega}^{-1})\\\leq&\delta x^T\left(\dot{M}+2\,\mathrm{sym}\left(Mf_x\right)+\alpha_g I\right)\delta x+C\underline{\omega}^{-1}.\end{aligned}\tag{7}$$

The relation $2ab\leq\varepsilon^{-1}a^2+\varepsilon b^2$, which holds for any $a,b\in\mathbb{R}$ and $\varepsilon>0$, is used with $a=\sqrt{2/\underline{\omega}}$ and $b=\sqrt{L_m}\|\delta z\|$ to obtain the second inequality. This reduces to $\mathscr{L}V\leq-2\alpha V+C\underline{\omega}^{-1}$ under the condition (3). The result (5) follows as in the proof of Theorem 1 in [8]. Note that there is a trade-off when using large ε, as it would decrease the steady-state error with small C in (5) but make the constraint (3) tighter. ■

We use the following lemma for convexification in Sec. IV.

Lemma 1: (3) and (4) are convex and equivalent to

$$\begin{bmatrix}-\dot{\bar{W}}+2\,\mathrm{sym}\left(f_x(x,t)\bar{W}\right)+2\alpha\bar{W} & \bar{W}\\ \bar{W} & -\frac{\nu}{\alpha_g}I\end{bmatrix}\preceq 0 \tag{8}$$

$$I\preceq\bar{W}\preceq\chi I,\ \forall x,t \tag{9}$$

where $\chi=\overline{\omega}/\underline{\omega}$, $\bar{W}=\nu W=\nu M^{-1}$, and $\nu=1/\underline{\omega}$.

Proof: Since $\nu=1/\underline{\omega}>0$ and $W\succ 0$, multiplying (3) by ν and then by W from both sides preserves matrix definiteness [22, pp. 114]. These operations with the Schur's complement lemma [22, pp. 28] yield (8). The rest follows as in the proof of Lemma 1 of [2]. Note that (3) and (4) are convex as (8) and (9) are linear matrix inequalities. ■

Remark 1: The result of Lemma 1 with the use of decision variables χ, $\tilde{W}$, and ν yield a semidefinite program for finding an optimal contraction metric, which is solvable by various computationally-efficient algorithms [23].

Finally, Lemma 2 introduces the generalized SDC form of dynamical systems to be exploited also in Sec. IV.

Lemma 2: Suppose $f:\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ and $B:\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^{n\times m}$ are continuously differentiable. Then $\forall t\in\mathbb{R}_{\geq 0}$, $x,x_d:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$, and $u_d:\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^n$, $\exists A(x,x_d,t)$ s.t. $A(x,x_d,t)(x-x_d)=f(x,t)+B(x,t)u_d(x_d,t)-f(x_d,t)-B(x_d,t)u_d(x_d,t)$, and one such A is given as follows:

$$A(x,x_d,t)=\int_0^1\bar{f}_x(cx+(1-c)x_d,t)dc \tag{10}$$

where $\bar{f}(q,t)=f(q,t)+B(q,t)u_d(x_d,t)$. We call A an SDC form when it is constructed to satisfy controllability and observability conditions (see Theorem 2 and Corollary 1).

Proof: This follows from the integral relation given as $\int_0^1(d\bar{f}(cx+(1-c)x_d,t)/dc)dc=\bar{f}(x,t)-\bar{f}(x_d,t)$. ■

III. Neural Stochastic Contraction Metrics

This section illustrates how to construct an NSCM using state samples $S=\{x_i\}_{i=1}^N$ and stochastic contraction metrics given by Theorem 1. The underlying idea is analogous to the one of the NCM [2], which gives an optimal contraction metric for nonlinear systems with bounded disturbances, but the NSCM explicitly accounts for unbounded stochastic disturbances. For the sake of simplicity, we denote the contraction metric both for state estimation and feedback control as X with $\underline{m}I\preceq X\preceq\overline{m}I$, i.e., $\underline{m}=\overline{\omega}^{-1}$, $\overline{m}=\underline{\omega}^{-1}$, $X=M$ for control, and $\underline{m}=\underline{\omega}$, $\overline{m}=\overline{\omega}$, $X=W$ for estimation.

A. Data Preprocessing

Since $X\succ 0$, where X is a contraction metric for control or estimation, it has a unique upper triangular matrix $Y\in\mathbb{R}^{n\times n}$ with positive diagonal entries s.t. $X=Y^TY$ [24, pp. 441]. We use the nonzero entries of Y, denoted as $\theta(x,t)\in\mathbb{R}^{n(n+1)/2}$, for y_i of (1) to reduce its output dimension [2].

B. Lipschitz Condition and Spectral Normalization (SN)

We utilize SN in Sec. II-A to guarantee the Lipschitz condition of Theorem 1 or Proposition 2 in Sec. IV.

Proposition 1: Let $\vartheta(x;W_{nn})$ be a neural network (1) to model $\theta(x,t)$ in Sec. III-A, and N_{units} be the number of neurons in its last layer. Also, let $W_{sn}=\{W_\ell\}_{\ell=1}^{L+1}$, where $W_\ell=(\Omega_\ell/\|\Omega_\ell\|)C_{nn}$ for $1\leq\ell\leq L$, and $W_\ell=\sqrt{\overline{m}}(\Omega_\ell/\|\Omega_\ell\|)/\sqrt{N_{\mathrm{units}}}$ for $\ell=L+1$. If $\exists C_{nn},L_m>0$ s.t.

$$\begin{aligned}&2\left\|\vartheta_{x_i}(x;W_{sn})\right\|\left\|\vartheta_{x_j}(x;W_{sn})\right\|\\&+2\|\vartheta(x;W_{sn})\|\|\vartheta_{x_ix_j}(x;W_{sn})\|\leq L_m,\ \forall i,j,x,\Omega\end{aligned}\tag{11}$$

then we have $\|\mathscr{X}\|\leq\overline{m}$ and $\|\mathscr{X}_{x_ix_j}\|\leq L_m$, $\forall x_i,x_j$, where $\mathscr{X}$ is the neural network model for the contraction metric $X(x,t)$. The latter inequality implies $\mathscr{X}_{x_i}$, $\forall i$ is indeed Lipschitz continuous with 2-norm Lipschitz constant L_m.

Proof: Let $\mathscr{Y}$ be the neural net model of Y in Sec. III-A. By definition of $X=Y^TY$ and θ, where X is the contraction metric, we have $\|\mathscr{X}\|\leq\|\mathscr{Y}\|^2\leq\|\mathscr{Y}\|_F^2=\|\vartheta\|^2$. Thus, the relation $\|\vartheta(x;W_{sn})\|\leq\sqrt{N_{\mathrm{units}}}\|W_{L+1}\|$ yields $\|\mathscr{X}\|\leq\overline{m}$ for $W_{L+1}=\sqrt{\overline{m}}(\Omega_{L+1}/\|\Omega_{L+1}\|)/\sqrt{N_{\mathrm{units}}}$. Also, differentiating $\mathscr{X}$ twice yields $\|\mathscr{X}_{x_ix_j}\|/2\leq\|\mathscr{Y}_{x_i}\|\|\mathscr{Y}_{x_j}\|+\|\mathscr{Y}\|\|\mathscr{Y}_{x_ix_j}\|\leq\|\vartheta_{x_i}\|\|\vartheta_{x_j}\|+\|\vartheta\|\|\vartheta_{x_ix_j}\|$, where the second inequality is due to $\|\mathscr{Y}\|\leq\|\mathscr{Y}\|_F=\|\vartheta\|$. Substituting W_{sn} gives (11). ■

Example 1: To see how Proposition 1 works, let us consider a scalar input/output neural net with one neuron at each layer in (1). Since we have $\|\vartheta(x;W_{sn})\|\leq\|W_{L+1}\|$, $\mathscr{X}\preceq\overline{m}I$ is indeed guaranteed by $\|W_{L+1}\|=\sqrt{\overline{m}}$. Also, we can compute the bounds $\|\vartheta_x(x;W_{sn})\|\leq\sqrt{\overline{m}}C_{nn}^L$ and $\|\vartheta_{xx}(x;W_{sn})\|\leq\|W_{L+1}\|C_{nn}^L(\sum_{\ell=1}^LC_{nn}^\ell)=\sqrt{\overline{m}}C_{nn}^{L+1}(C_{nn}^L-1)/(C_{nn}-1)$ using SN. Thus, (11) can be solved for C_{nn} by standard nonlinear equation solvers, treating $\overline{m}$ and L_m as given constants.

Remark 2: For non-autonomous systems, we can treat t or time-varying parameters $p(t)$ as another input to the neural network (1) by sampling them in a given parameter range of interest. For example, we could use $p=[x_d,u_d]^T$ for systems with a target trajectory. This also allows us to use adaptive control techniques [25], [26] to update an estimate of p.

IV. mCV-STEM Sampling of Contraction Metrics

We introduce the modified ConVex optimization-based Steady-state Tracking Error Minimization (mCV-STEM) method, an improved version of CV-STEM [8] for sampling the metrics which minimize an upper bound of the steady-state mean squared tracking error via convex optimization.

Remark 3: Due to its contraction-based formulation, combination properties [15] also apply to the NSCM framework. For example, contraction is preserved through hierarchical combination of estimation and control (i.e. output feedback control), or through time-delayed communications [14].

A. Incremental Stability Analysis of mCV-STEM

We utilize the general SDC parametrization with a target trajectory (10), which captures nonlinearity through $A(x,x_d,t)$ or through multiple non-unique A_i [5], resulting in global exponential stability if the pair (A,B) of (12) is uniformly controllable [5], [7]. Note that x_d and u_d can be viewed as extra inputs to the NSCM as in Remark 2, but we could use Corollary 2 as a simpler formulation which guarantees local exponential stability without using a target trajectory. Further extension to control contraction metrics, which use differential state feedback $\delta u = K(x,t)\delta x$ [9]–[13], could be considered for sampling the metric with global reference-independent stability guarantees, achieving greater generality at the cost of added computation. Similarly, while we construct an estimator with global stability guarantees using the SDC form as in (22), a more general formulation could utilize geodesics distances between trajectories [4]. We remark that these trade-offs would also hold for deterministic control and estimation design via NCMs [2].

1) mCV-STEM Control: Consider the following system with controller $u \in \mathbb{R}^m$ and perturbation $\mathscr{W}(t)$:

$$dx = (f(x,t)+B(x,t)u)dt + G_c(x,t)d\mathscr{W}(t) \quad (12)$$
$$dx_d = (f(x_d,t)+B(x_d,t)u_d(x_d,t))dt \quad (13)$$

where $B:\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^{n\times m}$, $G_c:\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^{n\times d}$, $\mathscr{W}(t)$ is a d-dimensional Wiener process, and $x_d:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ and $u_d:\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^m$ denote the target trajectory.

Theorem 2: Suppose $\exists g_c \in [0,\infty)$ s.t. $\|G_c(x,t)\|_F \leq g_c$, $\forall x,t$, and $\exists M(x,x_d,t)\succ 0$ s.t. M_{x_i}, $M_{x_{d,i}}$, $\forall i$ are Lipschitz with 2-norm Lipschitz constant L_m. Let u be designed as

$$u = u_d(x_d,t) - B(x,t)^T M(x,x_d,t)(x-x_d) \quad (14)$$
$$\dot{M} + 2\,\mathrm{sym}(MA) - 2MBB^TM + \alpha_{gc}I \preceq -2\alpha M \quad (15)$$
$$\overline{\omega}^{-1}I \preceq M(x,x_d,t) \preceq \underline{\omega}^{-1}I,\ \forall x,t \quad (16)$$

where $\alpha>0$, $\alpha_{gc} = L_m g_c^2(\varepsilon+1/2)$, $\varepsilon>0$, and A is given by (10) in Lemma 2. If the pair (A,B) is uniformly controllable, we have the following bound for the systems (12) and (13):

$$E[\|x-x_d\|^2] \leq \frac{C_c}{2\alpha}\chi + \overline{\omega}E[V(x(0),\delta q(0),0)]e^{-2\alpha t} \quad (17)$$

where $V(x,x_d,\delta q,t) = \delta q^T M(x,x_d,t)\delta q$, $C_c = g_c^2(2/\varepsilon+1)$, $\chi = \overline{\omega}/\underline{\omega}$, and $\nu = 1/\underline{\omega}$. Further, (15) and (16) are equivalent to the following linear matrix inequalities in terms of χ, ν, and $\bar{W} = \nu W = \nu M(x,x_d,t)^{-1}$:

$$\begin{bmatrix} -\dot{\bar{W}} + 2\,\mathrm{sym}(A\bar{W}) - 2\nu BB^T + 2\alpha\bar{W} & \bar{W} \\ \bar{W} & -\frac{\nu}{\alpha_{gc}}I \end{bmatrix} \preceq 0 \quad (18)$$
$$I \preceq \bar{W} \preceq \chi I,\ \forall x,t. \quad (19)$$

where the arguments are omitted for notational simplicity.

Proof: Using the SDC parameterization (10) given in Lemma 2, (12) can be written as $dx = (\bar{f}(x_d,t) + (A(x,x_d,t) - B(x,t)B(x,t)^TM(x,x_d,t))(x-x_d))dt + G_c(x,t)d\mathscr{W}$. This results in the following differential system, $dq = (\bar{f}(x_d,t) + (A(x,x_d,t) - B(x,t)B(x,t)^TM)(q-x_d))dt + G(q,t)d\mathscr{W}$, where $G(q,t)$ is defined as $G(x,t) = G_c(x,t)$ and $G(x_d,t) = 0$. Note that it has $q = x, x_d$ as particular solutions. Since f_x, g_1, and g_2 in Theorem 1 can be viewed as $A(x,x_d,t) - B(x,t)B(x,t)^TM(x,x_d,t)$, g_c, and 0, respectively, applying its results for $V = \delta q^T M(x,x_d,t)\delta q$ gives (17) as in (5). The constraints (18) and (19) follow from the application of Lemma 1 to (15) and (16). ■

Remark 4: For input non-affine nonlinear systems, we could find $\tilde{f}(x,x_d,u,t) = A(x,u,t)(x-x_d) + B(x,u,t)(u-u_d)$ by Lemma 2 and use it in Theorem 2, although (14) has to be solved implicitly as B depends on u in this case [12], [13].

2) mCV-STEM Estimation: Consider the following system and a measurement $y(t)$ with perturbation $\mathscr{W}_1(t)$ and $\mathscr{W}_2(t)$:

$$dx = f(x,t)dt + G_e(x,t)d\mathscr{W}_1(t) \quad (20)$$
$$ydt = h(x,t)dt + D(x,t)d\mathscr{W}_2(t) \quad (21)$$

where $h:\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^m$, $G_e:\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^{n\times d_1}$, $D:\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^{m\times d_2}$, and $W_{1,2}(t)$ are two independent Wiener processes. We have an analogous result to Theorem 2.

Corollary 1: Suppose $\exists g_e,\bar{d} \in [0,\infty)$ s.t. $\|G_e(x,t)\|_F \leq g_e$ and $\|D(x,t)\|_F \leq \bar{d}$, $\forall x,t$. Suppose also that $\exists W(\hat{x},t) = M(\hat{x},t)^{-1}\succ 0$ s.t. W_{x_i}, $\forall i$ is Lipschitz continuous with 2-norm Lipschitz constant L_m. Let $\nu = 1/\underline{\omega}$ and x be estimated as

$$d\hat{x} = f(\hat{x},t)dt + M(\hat{x},t)C_L(\hat{x},t)^T(y-h(\hat{x},t))dt \quad (22)$$
$$\dot{W} + 2\,\mathrm{sym}(WA - C_L^TC) + \alpha_{ge}I \preceq -2\alpha W \quad (23)$$
$$\underline{\omega}I \preceq W(\hat{x},t) \preceq \overline{\omega}I,\ 0<\nu\leq\sqrt[3]{\nu_c},\ \forall x,\hat{x},t \quad (24)$$

where $\alpha,\nu_c,\varepsilon>0$, $\alpha_{ge} = \alpha_{e1} + \nu_c\underline{\omega}\alpha_{e2}$, $\alpha_{e1} = L_m g_e^2(\varepsilon+1/2)$, and $\alpha_{e2} = L_m\bar{c}^2\bar{d}^2(\varepsilon+1/2)$. Also, $A(x,\hat{x},t)$ and $C(x,\hat{x},t)$ are given by (10) of Lemma 2 with (f,x,x_d,u_d) replaced by $(f,\hat{x},x,0)$ and $(h,\hat{x},x,0)$, respectively, and $C_L(\hat{x},t) = C(\hat{x},\hat{x},t)$. If (A,C) is uniformly observable and $\|C(x,\hat{x},t)\|\leq\bar{c}$, $\forall x,\hat{x},t$, then we have the following bound:

$$E[\|x-\hat{x}\|^2] \leq \frac{C_e}{2\alpha} + \overline{\omega}E[V(x(0),\delta q(0),0)]e^{-2\alpha t} \quad (25)$$

where $V(\hat{x},\delta q,t) = \delta q^T W(\hat{x},t)\delta q$, $C_e = C_{e1}\chi + C_{e2}\chi\nu^2$, $C_{e1} = g_e^2(2/\varepsilon+1)$, $C_{e2} = \bar{c}^2\bar{d}^2(2/\varepsilon+1)$, and $\chi = \overline{\omega}/\underline{\omega}$. Further, (23) and (24) are equivalent to the following convex constraints in terms of χ, ν, ν_c, and $\bar{W} = \nu W$:

$$\dot{\bar{W}} + 2\,\mathrm{sym}(\bar{W}A - \nu C_L^TC) + \alpha_{e1}I + \nu_c\alpha_{e2}I \preceq -2\alpha\bar{W} \quad (26)$$
$$I \preceq \bar{W} \preceq \chi I,\ 0<\nu\leq\sqrt[3]{\nu_c},\ \forall x,\hat{x},t \quad (27)$$

where the arguments are omitted for notational simplicity.

Proof: The differential system of (20) and (22) is given as $dq = f(x,t) + (A(x,\hat{x},t) - M(\hat{x},t)C_L(\hat{x},t)^TC(x,\hat{x},t))(q-x))dt + G(q,t)d\mathscr{W}$, where $G(q,t)$ is defined as $G(x,t) = G_e(x,t)$ and $G(\hat{x},t) = M(\hat{x},t)C(\hat{x},t)^TD(x,t)$. Viewing V, g_1, and g_2 in Theorem 1 as $V = \delta q^TW(\hat{x},t)\delta q$, $g_1 = g_e$, and $g_2 = \bar{c}\bar{d}/\underline{\omega}$, (25) – (27) follow as in the proof of Theorem 2 due to $\nu^3 = \underline{\omega}^{-3}\leq\nu_c$ and the contraction condition (23). ■

Note that (15) and (23) depend on their target trajectory, i.e. x_d for control and x for estimation. We can treat them as time-varying parameters $p(t)$ in a given space during the mCV-STEM sampling as in Remark 2. Alternatively, we could use the following to avoid this complication.

Corollary 2: Using predefined trajectories (e.g. $(x_d,u_d) = (0,0)$ for control or $x=0$ for estimation) in Thm. 2 or Cor. 1 leads to local exponential stability of (12) or (22).

Proof: This follows as in the proof of Thm. 2 [2]. ■

B. mCV-STEM Formulation

The following proposition summarizes the mCV-STEM.

Proposition 2: Suppose that α, ε, and L_m are given. The optimal contraction metric $M = W^{-1}$ that minimizes an upper bound of the steady-state mean squared distance of stochastically perturbed system trajectories can be found by the following convex optimization problem:

$$J_{CV}^* = \min_{\nu>0,\nu_c>0,\chi\in\mathbb{R},\bar{W}\succ 0} c_1\chi + c_2\nu + c_3 P(\nu,\nu_c,\chi,\bar{W}) \quad (28)$$

$$\text{s.t. (18) \& (19) for control, (26) \& (27) for estimation}$$

where $c_1, c_2, c_3 \in [0,\infty)$ and P is an additional performance-based convex cost function. The weight of ν, c_2, can either be viewed as a penalty on the 2-norm of feedback gains or an indicator of how much we trust the $y(t)$.

Proof: For control (17), using $c_1 = C_c/(2\alpha)$ and $c_2 = c_3 = 0$ gives (28). We can set $c_2 > 0$ to penalize excessively large $\|u\|$ through $\nu \geq \sup_{x,t}\|M(x,x_d,t)\|$. Since we have $\nu \geq 0$ and $1 \leq \chi \leq \chi^3$, (25) as $t \to \infty$ can be bounded as

$$\frac{C_{e1}\chi + C_{e2}\chi\nu^2}{2\gamma} \leq \frac{1}{3\sqrt{3C_{e1}}}\left(\frac{\sqrt{3C_{e1}}}{\sqrt[3]{2\gamma}}\chi + \frac{\sqrt{C_{e2}}}{\sqrt[3]{2\gamma}}\nu\right)^3. \quad (29)$$

Minimizing the right-hand side of (29) gives (28) with $c_1 = \sqrt{3C_{e1}}/\sqrt[3]{2\gamma}$, $c_2 = \sqrt{C_{e2}}/\sqrt[3]{2\gamma}$, and $c_3 = 0$. Finally, since $\bar{d} = 0$ in (21) means $C_{e2} = 0$ and no noise acts on y, c_2 also indicates how much we trust the measurement. ■

1) Choice of Objective Functions: Selecting $c_3 = 0$ in Proposition 2 yields an affine objective function which leads to a straightforward interpretation of its weights. Users could also select $c_3 > 0$ with other performance-based cost functions $P(\nu,\nu_c,\chi,\bar{W})$ in (28) as long as they are convex. For example, an objective function $\sum_{x_i\in S}\|u\|^2 = \sum_{x_i\in S}\|-B(x_i,t)^T M(x_i,t)x_i\|^2 \leq \sum_{x_i\in S}\|B(x_i,t)\|^2\|x_i\|^2\nu^2$, where S is the state space of interest, gives an optimal contraction metric which minimizes an upper bound of its control effort.

2) Additional Design Parameters: We assume α, ε, and L_m are given in Proposition 2. For α and ε, we perform line search to find their optimal values as will be demonstrated in Sec. V. For L_m, we could guess it by a deterministic NCM [2] and guarantee the Lipschitz condition by SN as explained in Sec. III-B. Also, (28) can be solved as a finite-dimensional problem by using backward difference approximation on W. We can then use $-\bar{W} \preceq -I$ to obtain a sufficient condition of its constraints, or solve it along pre-computed trajectories $\{x(t_i)\}_{i=0}^M$ [2], [27]. The pseudocode to obtain the NSCM depicted in Fig. 1 is given in Algorithm 1.

V. Numerical Implementation Example

We demonstrate the NSCM on a rocket autopilot problem (https://github.com/astrohiro/nscm). CVXPY [28] with the MOSEK solver [29] is used to solve convex optimization.

A. Simulation Setup

We use the nonlinear rocket model depicted in Fig. 2 [30], assuming that q and specific normal force measurements are available via rate gyros and accelerometers. We use $G_c = 6.0\times 10^{-2}I_n$, $G_e = 3.0\times 10^{-2}I_n$, and $D = 3.0\times 10^{-2}I_m$ for stochastic perturbation in the NSCM construction. The Mach number is assumed to be linearly time-varying from 2 to 4.

Algorithm 1: NSCM Algorithm

Inputs : States & parameters: $S = \{x_i\}_{i=1}^N$ or $\{\hat{x}_i\}_{i=1}^N$ & $T = \{p_i\}_{i=1}^M$ (e.g. $p = t$, $[x_d, u_d]^T$, or x)

Outputs: NSCM and J_{CV}^* in (28)

1. Sampling of Optimal Contraction Metrics

Find L_m in Thm. 1 using a deterministic NCM [2]

for $(\alpha,\varepsilon) \in A_{LS}$ **do**

  Solve (28) of Prop. 2 using x, p or $\hat{x}, p$ in S & T

  Save the optimizer $(\nu,\chi,\{\bar{W}_i\}_{i=1}^N)$ and optimal value $J(\alpha,\varepsilon) = c_1\chi + c_2\nu + c_3P(\nu,\nu_c,\chi,\bar{W})$

Find $(\alpha^*,\varepsilon^*) = \arg\min_{(\alpha,\varepsilon)\in A_{LS}} J$ and $J_{CV}^* = J(\alpha^*,\varepsilon^*)$

Obtain $(\nu(\alpha^*,\varepsilon^*),\chi(\alpha^*,\varepsilon^*),\{\bar{W}_i(\alpha^*,\varepsilon^*)\}_{i=1}^N)$

2. Spectrally-Normalized Neural Network Training

Preprocess data as in Sec. III-A

Split data into a train set $\mathscr{S}_{\text{train}}$ and test set $\mathscr{S}_{\text{test}}$

for epoch $\leftarrow 1$ **to** N_{epochs} **do**

  for $s \in \mathscr{S}_{\text{train}}$ **do**

    Train the neural network using SGD with the Lipschitz condition on X_{x_i} as in Prop. 1

  Compute the test error for data in $\mathscr{S}_{\text{test}}$

  if test error is small enough **then**

    break

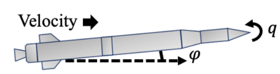

Fig. 2. Rocket model (angle of attack φ, pitch rate q).

B. NSCM Construction

We construct NSCMs using Algorithm 1. For estimation, we select the Lipschitz constant on X_{x_i} to be $L_m = 0.50$ as discussed in Sec. IV-B2. The optimal α and ε are determined to be $\alpha = 0.40$ and $\varepsilon = 2.40$ by line search in Fig. 3. A neural network with 3 layers and 100 neurons is trained using $N = 1000$ samples, where its SN constant is selected as $C_{nn} = 0.85$ as a result of Proposition 1. We use the same approach for the NSCM control and the resultant design parameters are given in Table I. Figure 4 implies that the NSCMs indeed satisfy the Lipschitz condition with its prediction error smaller than 0.03 thanks to the SN technique.

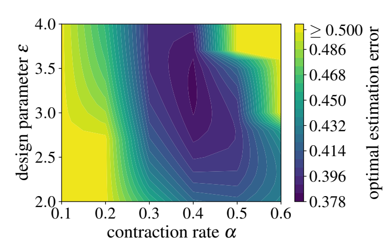

Fig. 3. Optimal steady-state estimation error as a function of α and ε.

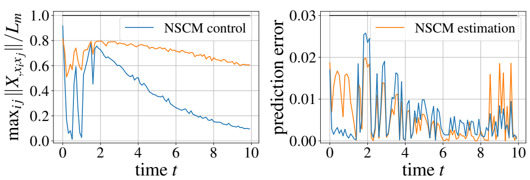

Fig. 4. NSCM spectral normalization and prediction error.

TABLE I
NSCM CONTROL AND ESTIMATION PARAMETERS

	α	ε	L_m	steady-state upper bound
estimation	0.40	2.40	0.50	0.39
control	0.10	1.00	10.00	0.58

VI. DISCUSSION AND CONCLUDING REMARKS

We compare the NSCM with the SDRE [7], ILQR [16], [17], EKF, NCM [2], and mCV-STEM. As shown in Fig. 5, the steady-state errors of the NSCM and mCV-STEM are indeed smaller than its steady-state upper bound in Proposition 2, while the others violate this condition. Also, the optimal contraction rate of the NCM for state estimation is much larger ($\alpha = 6.1$) than the NSCM as it does not account for stochastic perturbation. This renders the NCM trajectory diverge around $t = 5.8$ in Fig. 5. The NSCM Lipschitz condition on X_{x_i} guaranteed by SN as in Fig. 4 allows us to circumvent this difficulty.

In conclusion, the NSCM is a novel way of using spectrally-normalized neural networks for real-time computation of approximate nonlinear control and estimation policies, which are optimal and provably stable in the mean squared error sense even under stochastic disturbances. We remark that the reference-independent policies [4], [9]–[13] or the generalized SDC policies (14) and (22) introduced in this paper, which guarantee global exponential stability with respect to a target trajectory, could be used both in stochastic and deterministic frameworks including the NCM [2]. It is also noted that the combination properties of contraction theory in Remark 3 still holds for the deterministic NCM.

VII. CONCLUSION

This paper presents an NSCM, a spectrally-normalized neural network model of an optimal stochastic contraction metric. Thanks to spectral normalization, which constrains the first state-derivatives of the metric to be Lipschitz continuous, this framework enables real-time computation of approximate control and estimation policies optimal and provably stable in a mean squared error sense even under stochastic disturbances. It outperforms existing nonlinear estimation and control techniques as shown in the simulation.

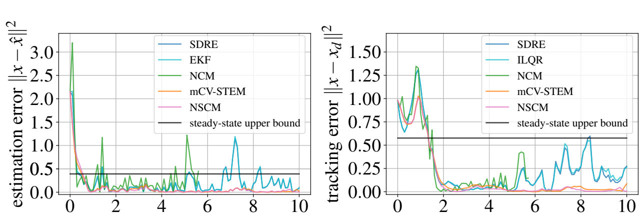

Fig. 5. Rocket state estimation and tracking error ($x = [\varphi, q]^T$).

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