Statistical Contraction Theory for Learning-based Control of Nonlinear Systems—Part 2: Robustness & Control

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- In the following, $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced 2-norm for matrices, i.e., $\|\cdot\| = \|\cdot\|_2$, unless otherwise noted.
- For a symmetric square matrix $S \in \mathbb{R}^{n\times n}$, we use the notation $S \succ 0$, $S \succeq 0$, $S \prec 0$, and $S \preceq 0$ for the positive definite, positive semi-definite, negative definite, and negative semi-definite matrices, respectively.
- This lecture is based on [1, 2, 3, 4].

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Robustness in Contraction Theory

Let's now consider the following perturbed dynamical system:

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) + d(\mathbf{x}, t) \tag{1}$$

where $d(\mathbf{x}, t)$ represents external disturbance with $\sup_{\mathbf{x},t} \|d(\mathbf{x}, t)\| = \bar{d} < \infty$.

Robustness in Contraction Theory

Let

(1) $\mathbf{x}_1$ be an arbitrary perturbed solution trajectory of (1)

(2) $\mathbf{x}_0$ be an arbitrary unperturbed solution trajectory of (1) with $d(\mathbf{x}, t) = 0$

(3) $\mathbf{q}(\mu, t)$ be a smooth path connecting $\mathbf{x}_0$ and $\mathbf{x}_1$, parameterized by $\mu \in [0, 1]$, with $\mathbf{q}(1, t) = \mathbf{x}_1$ and $\mathbf{q}(0, t) = \mathbf{x}_0$.

Then we can define its virtual system as follows:

$$\dot{\mathbf{q}} = f(\mathbf{q}, t) + \mu d(\mathbf{x}_1, t) \tag{2}$$

which indeed gives

(1) $\dot{\mathbf{x}}_1 = f(\mathbf{x}_1, t) + \mu d(\mathbf{x}_1, t)$ if $\mu = 1$

(2) $\dot{\mathbf{x}}_0 = f(\mathbf{x}_0, t)$ if $\mu = 0$.

The differential dynamics is given by

$$\frac{d}{dt}\left(\frac{\partial \mathbf{q}}{\partial \mu}\right) = \frac{\partial f}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial \mu} + d(\mathbf{x}_1, t).$$

Robustness in Contraction Theory

Theorem (Robust Contraction)

Suppose that there exists $M(\mathbf{x},t)=\Theta(\mathbf{x},t)^\top\Theta(\mathbf{x},t)\succ 0$, $\forall \mathbf{x},t$, s.t. the following conditions hold for $\exists\alpha>0$, $\forall \mathbf{x},t$:

$$\dot{M}(\mathbf{x},t)+M(\mathbf{x},t)\frac{\partial f}{\partial \mathbf{x}}+\frac{\partial f}{\partial \mathbf{x}}^\top M(\mathbf{x},t)\preceq -2\alpha M(\mathbf{x},t)\ \&\ \underline{m}\mathbb{I}\preceq M(\mathbf{x},t)\preceq \overline{m}\mathbb{I}$$

where $\mathbb{I}$ is the identity matrix. Then we have the following:

$$d_M(t)\leq d_M(0)e^{-\alpha t}+\frac{\bar{d}\sqrt{\overline{m}}}{\alpha}(1-e^{-\alpha t}) \tag{3}$$

where $d_M(t)=\int_0^1\left\|\Theta(\mathbf{q}(\mu,t),t)\frac{\partial \mathbf{q}(\mu,t)}{\partial \mu}\right\|d\mu$.

Robustness in Contraction Theory

Note that more intuitively, (3) reduces to

$$\|\mathbf{x}_1(t) - \mathbf{x}_0(t)\| \leq \frac{d_M(0)}{\sqrt{\underline{m}}} e^{-\alpha t} + \frac{\bar{d}}{\alpha}\sqrt{\frac{\overline{m}}{\underline{m}}}(1 - e^{-\alpha t})$$

for $\mathbf{x}_1$ and $\mathbf{x}_0$ defined in (2).

Robustness in Contraction Theory

Proof: Let $V = \frac{\partial \mathbf{q}}{\partial \mu}^{\top} M(\mathbf{q},t)\frac{\partial \mathbf{q}}{\partial \mu}$. Using the contraction condition and the virtual dynamics (2), we get

$$\dot{V} \leq -2\alpha V + 2\frac{\partial \mathbf{q}}{\partial \mu}^{\top} M(\mathbf{q},t) d(\mathbf{x}_1,t) \leq -2\alpha V + 2\bar{d}\sqrt{m}\left\|\Theta(\mathbf{q},t)\frac{\partial \mathbf{q}}{\partial \mu}\right\|.$$

Since $V = \left\|\Theta(\mathbf{q},t)\frac{\partial \mathbf{q}}{\partial \mu}\right\|^2$ by definition, we get

$$\frac{d}{dt}\left(\left\|\Theta(\mathbf{q},t)\frac{\partial \mathbf{q}}{\partial \mu}\right\|\right) \leq -\alpha\left\|\Theta(\mathbf{q},t)\frac{\partial \mathbf{q}}{\partial \mu}\right\| + \bar{d}\sqrt{m}.$$

as long as $\left\|\Theta(\mathbf{q},t)\frac{\partial \mathbf{q}}{\partial \mu}\right\| \neq 0$. Applying the comparison lemma and then path-integrating the results yields (3). □

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A Linear System as an Analogy

Let's start with a very simple example. Consider a linear dynamical system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the system state, $\mathbf{u} \in \mathbb{R}^m$ is the control input, and A and B are constant matrices of appropriate dimensions.

A Linear System as an Analogy

How can we design $\mathbf{u}$ for (4) to exponentially track an arbitrary desired trajectory $\mathbf{x}_d$ given by $\dot{\mathbf{x}}_d = A\mathbf{x}_d + B\mathbf{u}_d$, where $\mathbf{u}_d$ is an arbitrary desired trajectory?

A Linear System as an Analogy

We often use the linear quadratic regulator (LQR) given by

$$\mathbf{u} = \mathbf{u}_d - K(\mathbf{x} - \mathbf{x}_d)$$

where $K = R^{-1}B^{\top}P$ is the control gain with the weight R.

A Linear System as an Analogy

For a quadratic Lyapunov function candidate $V = (\mathbf{x} - \mathbf{x}_d)^\top P(\mathbf{x} - \mathbf{x}_d)$ with $P \succ 0$, the Lyapunov decreasing condition is given by

$$\dot{V} = (\mathbf{x} - \mathbf{x}_d)^\top (PA + A^\top P - 2PBR^{-1}B^\top P)(\mathbf{x} - \mathbf{x}_d) \leq -2\alpha(\mathbf{x} - \mathbf{x}_d)^\top P(\mathbf{x} - \mathbf{x}_d)$$

where $\alpha > 0$.

A Linear System as an Analogy

This gives the following matrix inequality (Riccati inequality):

$$PA + A^\top P - 2PBR^{-1}B^\top P \preceq -2\alpha P$$

which can be convexified as

$$AQ + QA^\top - 2BR^{-1}B^\top \preceq -2\alpha Q$$

where $Q = P^{-1}$.

A Linear System as an Analogy

Can we formulate an analogous result for nonlinear systems?

A Linear System as an Analogy

Can simply use $V = \delta \mathbf{x}^\top M(\mathbf{x}, t) \delta \mathbf{x}$ instead of $V = (\mathbf{x} - \mathbf{x}_d)^\top P(\mathbf{x} - \mathbf{x}_d)$?

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Differential Feedback Control

Let's now consider the following nonlinear dynamical system:

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$. The differential dynamics is given by

$$\delta\dot{\mathbf{x}} = \frac{\partial f}{\partial \mathbf{x}}\delta\mathbf{x} + B(\mathbf{x}, t)\delta\mathbf{u} + \sum_{i=1}^{n} \frac{\partial b_i}{\partial \mathbf{x}}\delta\mathbf{x} \cdot u_i$$

where u_i is the ith element of $\mathbf{u}$ and $b_i(\mathbf{x}, t)$ is the ith column of $B(\mathbf{x}, t)$.

Differential Feedback Control

As we did in LQR, consider the following differential feedback control policy:

$$\delta \mathbf{u} = -K(\mathbf{x}, t)\delta \mathbf{x} \tag{5}$$

where $K(\mathbf{x}, t) = R(\mathbf{x}, t)^{-1} B(\mathbf{x}, t)^\top M(\mathbf{x}, t)$ is again the control gain with the state- and time-dependent weight $R(\mathbf{x}, t)$.

Differential Feedback Control

Note that given arbitrary desired control policy $\mathbf{u}_d$ and $\mathbf{x}_d$, $\mathbf{u}$ can be computed by path-integrating (5).

Differential Feedback Control

Then we can show that the following conditions with $M \succ 0$ and $\alpha > 0$

$$\frac{\partial M}{\partial t} + \sum_{k=1}^{n} \frac{\partial M}{\partial x_k} f_k + M \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial f}{\partial \mathbf{x}}^\top M - 2MBR^{-1}B^\top M \preceq -2\alpha M$$
$$\sum_{k=1}^{n} \frac{\partial M}{\partial x_k} b_{i,k} + M \frac{\partial b_i}{\partial \mathbf{x}} + \frac{\partial b_i}{\partial \mathbf{x}}^\top M = 0, \ \forall i = 1, \cdots, m \tag{6}$$

yield $\dot{V} \leq -2\alpha V$, where $V = \delta \mathbf{x}^\top M(\mathbf{x}, t) \delta \mathbf{x}$, x_k is the kth element of $\mathbf{x}$, $f_k(\mathbf{x}, t)$ is the kth element of $f(\mathbf{x}, t)$, and $b_{i,k}(\mathbf{x}, t)$ is the kth element of $b_i(\mathbf{x}, t)$ (HW).

Differential Feedback Control

Note that the conditions (6) can be convexified as follows (HW):

$$-\frac{\partial W}{\partial t}-\sum_{k=1}^{n}\frac{\partial W}{\partial x_k}f_k+\frac{\partial f}{\partial \mathbf{x}}W+W\frac{\partial f}{\partial \mathbf{x}}^{\top}-2BR^{-1}B^{\top}\preceq -2\alpha W$$

$$-\sum_{k=1}^{n}\frac{\partial W}{\partial x_k}b_{i,k}+\frac{\partial b_i}{\partial \mathbf{x}}W+W\frac{\partial b_i}{\partial \mathbf{x}}^{\top}=0,\ \forall i=1,\cdots,m$$

where $W=M^{-1}$. Note that we can further simplify these conditions by using the geodesic (see [3, 4] for details).

Reference

[1] W. Lohmiller and J.-J. E. Slotine. "On contraction analysis for nonlinear systems". In: *Automatica* 34.6 (1998), pp. 683–696. ISSN: 0005-1098. DOI: 10.1016/S0005-1098(98)00019-3.

[2] H. Tsukamoto and S.-J. Chung. "Robust controller design for stochastic nonlinear systems via convex optimization". In: *IEEE Transactions on Automatic Control* 66.10 (2021), pp. 4731–4746. DOI: 10.1109/TAC.2020.3038402.

[3] I. R. Manchester and J.-J. E. Slotine. "Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design". In: *IEEE Transactions on Automatic Control* 62.6 (June 2017), pp. 3046–3053. DOI: 10.1109/TAC.2017.2668380.

Reference

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Controller Design—A Linear System as an Analogy

Controller Design with Contraction Theory

References

[4] H. Tsukamoto, S.-J. Chung, and J.-J. E. Slotine. "Contraction theory for nonlinear stability analysis and learning-based control: A tutorial overview". In: *Annual Reviews in Control* 52 (2021), pp. 135–169. ISSN: 1367-5788. DOI: 10.1016/S0005-1098(98)00019-3.